

Transient Response of a Dipole Antenna*

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The current distributed for a dipole antenna driven by a step-function voltage is found shortly after the switch-on of the voltage.

ALTHOUGH many approximate methods have been devised to deal with the dipole antenna of finite length, no case has been solved exactly in a form appropriate for computation insofar as the author is aware. In this paper, the initial behavior of the transient response of a model of the dipole antenna is considered. If h is the half-length of the antenna, then for $x_0 = ct < h$ the dipole behaves as though it were infinitely long and the problem can be solved exactly. The model of the dipole antenna which we shall adopt is a perfectly conducting, infinitely thin tube located at $r = a$, in our cylindrical coordinate system r, θ, x . The external voltage is applied at $r = a, x = 0$:

$$E_x^{\text{ext}} = \begin{cases} \delta(x) & t > 0, \\ 0 & t < 0. \end{cases} \quad (1)$$

The problem is rotationally symmetrical with respect to θ .

Let G be the retarded Green's function for the scalar wave equation

$$(\nabla^2 - \partial^2/\partial x_0^2)G(\mathbf{r}, x_0) = -c\delta(\mathbf{r})\delta(x_0), \quad (2)$$

then the total current $\hat{x}I(x, x_0)$ on the dipole induces the vector potential

$$\mathbf{A}(\mathbf{r}, x_0) = \mu_0 \hat{x} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' I(x', x_0') (2\pi)^{-1} \times \int_0^{2\pi} d\theta' (\mathbf{r} - \mathbf{r}', x_0 - x_0'). \quad (3)$$

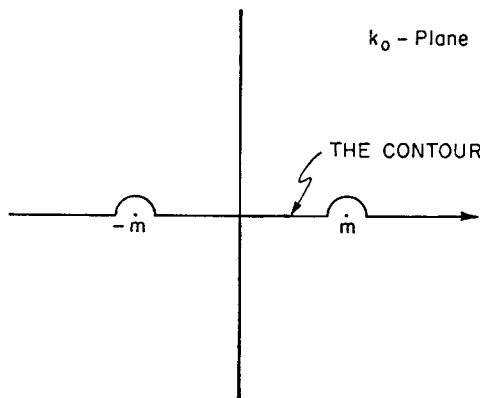


FIG. 1. Projection of C_0 on k_0 plane.

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On the other hand, it follows from (1) that for $r = a$, $\mathbf{A}$ satisfies

$$(\partial^2/\partial x^2 - \partial^2/\partial x_0^2)\mathbf{A} \cdot \hat{x} = -c^{-1}\delta(x)\delta(x_0). \quad (4)$$

Since G is explicitly given by

$$G(\mathbf{r}, x_0) = (4\pi x_0)^{-1} c \delta(r - x_0), \quad (5)$$

(3) and (4) lead to the integral equation for I

$$(\partial^2/\partial x^2 - \partial^2/\partial x_0^2) \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx_0' I(x', x_0') \times K(x - x', x_0 - x_0') = -4\pi \zeta_0^{-1} \delta(x)\delta(x_0), \quad (6)$$

where ζ_0 is the characteristic impedance of free space, and the kernel K is

$$K(x, x_0) = (2\pi x_0)^{-1} \int_{-\pi}^{\pi} d\theta \delta([x^2 + (2a \sin\theta/2)^2]^{1/2} - x_0). \quad (7)$$

Equation (6) may be solved exactly by a Fourier transformation with respect to both variables. In general, it is very hard to compute numerically the value of a double Fourier transform. However, in the present case, the double Fourier transform may be reduced to a single integral due to the invariance of (6) under the formal one-dimensional orthochronous Lorentz transformation

$$\begin{aligned} \mathcal{L}x &= x \cosh\phi + x_0 \sinh\phi, \\ \mathcal{L}x_0 &= x \sinh\phi + x_0 \cosh\phi. \end{aligned} \quad (8)$$

To make use of this invariance, introduce a formal "photon mass" $m > 0$, and consider (6) to be the limit as $m \rightarrow 0+$ of

$$(\partial^2/\partial x^2 - \partial^2/\partial x_0^2 - m^2) \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx_0' I(x', x_0') \times K(x - x', x_0 - x_0') = -4\pi \zeta_0^{-1} \delta(x)\delta(x_0). \quad (9)$$

The Green's function g for the one-dimensional Klein-Gordon equation

$$(\partial^2/\partial x^2 - \partial^2/\partial x_0^2 - m^2)g(x, x_0) = -\delta(x)\delta(x_0) \quad (10)$$

has the representation

$$g(x, x_0) = -(2\pi)^{-2} \int_{s_0} dk dk_0 (k_0^2 - k^2 - m^2)^{-1} \times \exp[i(kx - k_0 x_0)]. \quad (11)$$

It is desired to choose the complex surface of integration S_0 so that g of (11) is the retarded Green's function and also that S_0 is invariant under the formal Lorentz transformation in the momentum space.

$$\begin{aligned}\mathcal{L}k &= k \cosh\phi + k_0 \sinh\phi, \\ \mathcal{L}k_0 &= k \sinh\phi + k_0 \cosh\phi.\end{aligned}\quad (12)$$

Let C_0 be a contour where $k=0$ and k_0 ranges from $-\infty$ to ∞ with detours near $k_0=\pm m$ as shown in Fig. 1. Note that C_0 is required to go through the origin in k_0 . Then a possible choice for S_0 is:

$$\begin{aligned}S_0: k, k_0 \text{ both real if } |k| > |k_0| \\ S_0 = \mathcal{L}C_0 \text{ if } |k_0| > |k|.\end{aligned}$$

The kernel K has the presentation

$$K(x, x_0) = (2\pi)^{-2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk_0 L(k, k_0) \times \exp[i(kx - k_0x_0)], \quad (13)$$

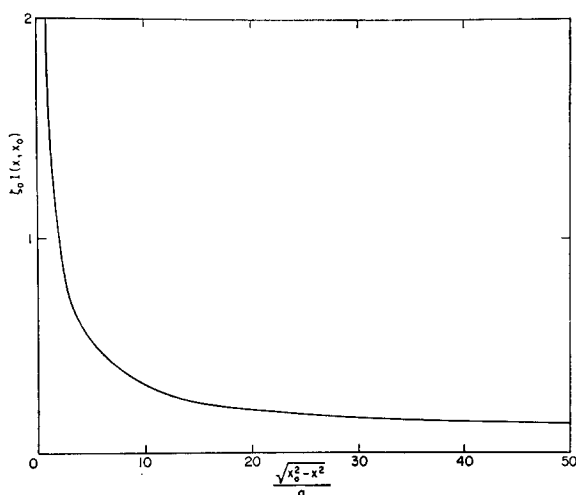


FIG. 2. Current distribution as a function of $(x_0^2 - x^2)^{1/2}/a$.

where

$$L(k, k_0) = \begin{cases} \pi i H_0^{(1)}[a(k_0^2 - k^2)^{1/2}] J_0[a(k_0^2 - k^2)^{1/2}], & \text{for } k_0 > |k|, \\ -\pi i H_0^{(2)}[a(k_0^2 - k^2)^{1/2}] J_0[a(k_0^2 - k^2)^{1/2}], & \text{for } k_0 < -|k|, \\ 2K_0[a(k^2 - k_0^2)^{1/2}] I_0[a(k^2 - k_0^2)^{1/2}], & \text{for } |k_0| < |k|. \end{cases} \quad (14)$$

Thus the solution of (9) is

$$I(x, x_0) = (2\pi)^{-1} \int_{S_0} dk dk_0 J(k, k_0) \exp[i(kx - k_0x_0)], \quad (15)$$

where

$$J(k, k_0) = -4\pi \zeta_0^{-1} (k_0^2 - k^2 - m^2)^{-1} [L(k, k_0)]^{-1}. \quad (16)$$

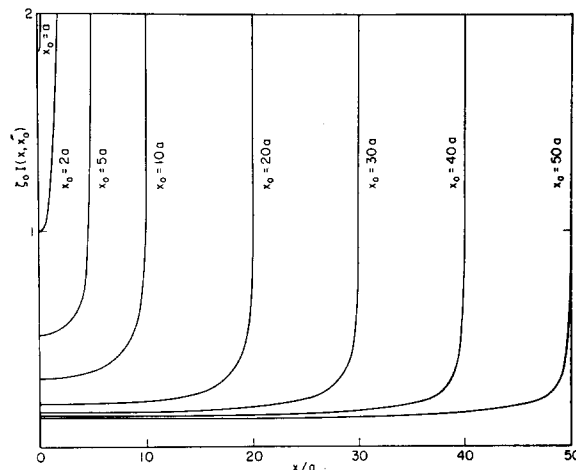


FIG. 3. Current distribution as a function of x for fixed x_0 .

As shown in the Appendix, (15) may be reduced to, in the limit $m \rightarrow 0+$,

$$I(x, x_0) = 4(\pi \zeta_0)^{-1} \int_0^\infty \frac{d\zeta}{\zeta} \frac{J_0[\zeta(x_0^2 - x^2)^{1/2}/a]}{[J_0(\zeta)]^2 + [Y_0(\zeta)]^2} \quad (17)$$

for $x_0 > x$ and is zero otherwise. For the purpose of numerical computation it may be advantageous to use the alternative form

$$I(x, x_0) = 2\pi \zeta_0^{-1} \int_0^\infty \frac{d\zeta}{\zeta} \frac{K_0[\zeta(x_0^2 - x^2)^{1/2}/a] I_0(\zeta)}{K_0(\zeta) \{ [K_0(\zeta)]^2 + \pi^2 [I_0(\zeta)]^2 \}}. \quad (18)$$

It is seen from (17) that, as $x_0^2 - x^2 \rightarrow 0+$,

$$I(x, x_0) \sim 2a \zeta_0^{-1} (x_0^2 - x^2)^{-1/2}. \quad (19)$$

For $x=0$, the singularity at $x_0=0$ is x_0^{-1} . This is not integrable and is responsible for the logarithmic singularity previously found for harmonic time dependence at $x=0$.¹ On the other hand, in the limit $x_0^2 - x^2 \rightarrow \infty$, various approximate techniques may be tried.

The behavior of the integral on the right-hand side of (18) is shown in Fig. 2. This curve contains all the information about the current distribution. In particular, it is straightforward to read off the dependence of I on x for various fixed values of x_0 . This is shown in Fig. 3.

Two remarks may be added. First, the transient response of the dipole as found is very different from that of a lossy transmission line excited in the same manner. Secondly, the response to a rectangular pulse may be obtained by superimposing two oppositely directed step functions separated by a finite interval of time.

¹ T. T. Wu and R. W. P. King, J. Appl. Phys. 30, 74 (1959).

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APPENDIX

In order to derive (17) from (15), let the surface S_0 be divided into four pieces:

- (1) $\text{Re} k_0 > |k|$; (2) $k > |k_0|$;
(3) $-\text{Re} k_0 > |k|$; and (4) $-k > |k_0|$.

Let I_i , $i=1\cdots 4$, be the contribution to I of (15) from these four regions, respectively. With the variable $(k_0^2 - k^2)^{\frac{1}{2}}$, it may be verified that

$$I_i(x, x_0) = -(\pi \zeta_0)^{-1} \int_{C_i} \zeta d\zeta [(\zeta^2 - m^2) \times \pi i H_0^{(1)}(a\zeta) J_0(a\zeta)]^{-1} F_i(x, x_0; \zeta). \quad (\text{A1})$$

In (A1), the four contours are as follows: C_1 is from 0 to ∞ along the positive real axis except for an upward detour near $\zeta = m$; C_2 and C_4 are both from $i\infty$ to 0 along the positive imaginary axis; and C_3 is from 0 to $-\infty$ along the negative real axis except for an upward detour near $\zeta = -m$. The functions F_i are explicitly

$$F_1(x, x_0; \zeta) = \begin{cases} -\pi i H_0^{(2)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } x > |x_0|, \\ \pi i H_0^{(1)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } -x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } -x > |x_0|, \end{cases} \quad (\text{A2})$$

$$F_2(x, x_0; \zeta) = \begin{cases} \pi i H_0^{(1)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}] + 2\pi i I_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } x > |x_0|, \\ \pi i H_0^{(1)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } -x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } -x > |x_0|, \end{cases} \quad (\text{A3})$$

$$F_3(x, x_0; \zeta) = \begin{cases} \pi i H_0^{(1)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}] + 2\pi i J_0[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}] + 2\pi i I_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } x > |x_0|, \\ \pi i H_0^{(1)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } -x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}] + 2\pi i I_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } -x > |x_0|, \end{cases} \quad (\text{A4})$$

and

$$F_4(x, x_0; \zeta) = \begin{cases} \pi i H_0^{(1)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } x > |x_0|, \\ \pi i H_0^{(1)}[\zeta(x_0^2 - x^2)^{\frac{1}{2}}], & \text{for } -x_0 > |x|, \\ 2K_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}] + 2\pi i I_0[\zeta(x^2 - x_0^2)^{\frac{1}{2}}], & \text{for } -x > |x_0|. \end{cases} \quad (\text{A5})$$

From (A2)–(A5) it is seen that $I_1 + I_4 = I_2 + I_3 = 0$ for $x_0 < x$ and $I_1 + I_2 = I_3 + I_4 = 0$ for $x_0 < -x$. Thus $I = 0$ unless $x_0 > |x|$. For $x_0 > |x|$, (A2)–(A5) give

$$I(x, x_0) = 2(\pi \zeta_0)^{-1} \int \zeta d\zeta (\zeta^2 - m^2)^{-1} J_0[\zeta(x_0^2 - x^2)^{\frac{1}{2}}] \times [H_0^{(1)}(a\zeta) J_0(a\zeta)]^{-1} \quad (\text{A6})$$

where the contour of integration is that of Fig. 1. Equation (17) now follows readily.